# Module 2: Basic Concepts of Trigonometric Functions

### II. Trigonometric Functions of Any Angle

After completing this section, you should be able to:

* determine the trigonometric function values of any angle in standard position
* use reference triangles, symmetry principles, and reference angles as aids in finding trigonometric function values

#### A. Angles in Standard Position

The notion of a trigonometric function can be extended to angles which are not acute angles. To describe a general angle, the angle is put in a frame of reference called standard position.

|  |  |
| --- | --- |
| An angle consists of two rays having the same endpoint, called the vertex.  One of the rays is placed along the positive x-axis with the vertex at the origin. This ray is called the initial side of the angle.  The other ray is called the terminal side of the angle.  This angle is said to be in standard position. |  |
| An angle can also be described in terms of rotation. Consider a point on the initial side of an angle. In order to get to the corresponding point on the terminal side of the angle and keep the vertex fixed, rotate the initial side until it coincides with the terminal side. |  |

There are two possible directions to take. The initial side could be rotated clockwise until it arrives at the terminal side, or the initial side could be rotated counterclockwise.

In the realm of mathematics, rotations are typically counterclockwise. A counterclockwise rotation is called a positive rotation, and the corresponding angle has a positive measure. A clockwise rotation is called a negative rotation, and the corresponding angle has a negative measure.

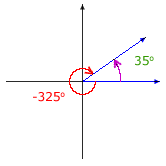
|  |  |  |
| --- | --- | --- |
| positive rotation |  | negative rotation |

A circle has 360 degrees. For a rotation of 360°, the terminal side and the initial side of the angle coincide.

Here are illustrations of some selected angles described in terms of rotation:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

The same angle may be described as a rotation in many ways.



The angle pictured can be described as a rotation of 35° or a rotation of –325°. Furthermore, since a complete rotation has 360 degrees, the angle can also be described as a rotation of 35° + 360° = 395°, or 35° + 2(360°) = 755°, or 35° – 2(360°) = –685°, and so on.

Adding or subtracting a multiple of 360° to a rotation results in the same angle in standard position. A given angle in standard position has an associated family of infinitely many rotations, and the measures of any two rotations in the family differ by a multiple of 360°. The members of the family are known as coterminal angles because they have the same terminal side even though their rotations have different measures.

**Example II.A.1:** Find two positive angles and two negative angles that are coterminal with the angle 160°.

**Solution:**

To find positive angles that are coterminal with 160°, add multiples of 360° to 160°:

One positive angle coterminal with 160° has measure 160° + 360° = 520°.  
Another has measure 160° + 2(360°) = 880°.

To find negative angles that are coterminal with 160°, subtract multiples of 360° from 160°:

One negative angle coterminal with 160° has measure 160° – 360° = –200°.  
Another negative angle coterminal with 160° has measure 160° – 2(360°) = –560°.

#### B. Reference Triangles and Trigonometric Function Values

For any angle in standard position that is not a multiple of 90°, it is possible to draw an associated right triangle called a reference triangle. The reference triangle is used to define trigonometric function values for the angle.

To determine the reference triangle, follow these steps:

**Example II.B.1:** Given an angle of 30°, draw the reference triangle and find the trigonometric function values sin 30°, cos 30°, and tan 30°.

**Solution:**

|  |  |
| --- | --- |
| Place the 30° angle in standard position. |  |
| Recall the 1-square root of negative 3-2 triangle associated with a 30° angle. The hypotenuse has length 2, so it is convenient to choose 2 for the radius of a circle.  Draw a circle of radius 2, and then draw the legs of the reference triangle. |  |
| For a 1-square root of negative 3-2 triangle, the side opposite 30° has length 1 and the adjacent side has length square root of negative 3.  The point P has coordinates (x, y) = (square root of negative 3, 1).  Using the coordinates of the point P and the radius r:  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/mod2-ex2-sinformula.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/mod2-ex2-cosformula.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/mod2-ex2-tanformula.gif | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/B-6-RefTri1-sqrt3-2a.png |

Do the trigonometric function values depend on the size of the circle? What do you think?

[Answer](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/trigfuncValue.html)

Not surprisingly, the trigonometric function values of 30° found using the reference triangle approach do agree with the values determined in the previous section on right triangle trigonometry! However, as you will soon see, this new approach will allow to you to find trigonometric function values for angles that are not necessarily acute.

The 30° angle has its terminal side in the first quadrant. (See module 1, topic I-A for a review of the four quadrants.) The trigonometric functions of angles in other quadrants can be defined using the same process carried out for the 30° angle.

# Trigonometric Functions of Any Angle

Suppose θ is an angle in standard position and P:(x, y) is a point on the terminal side, located a distance r from the origin. The trigonometric functions of the angle θ are defined in terms of x, y, and r:

|  |  |  |
| --- | --- | --- |
|  |  |  |

The cosecant, secant, and cotangent of θ are the reciprocals of the sine, cosine, and tangent of θ, respectively.

To find the trigonometric function values of an angle with terminal side in quadrants II, III, or IV, create a reference triangle and determine the ratios. Be careful to note that x or y may be negative, depending on the quadrant. The value r is always positive because it is a measure of distance. Reference triangles associated with angles in quadrants II, III, and IV are shown below.

|  |  |
| --- | --- |
| **Angle**θ**in Quadrant II x < 0, y > 0** | **Angle**θ**in Quadrant I x > 0, y > 0** |
|  |  |
| **Angle**θ**in Quadrant III x < 0, y < 0** | **Angle**θ**in Quadrant IV x > 0, y < 0** |
|  |  |

In practice, to find the trigonometric function values of a particular angle, start with either

* the measure of the angle

or

* a particular point P on the terminal side of the angle.

In example II.B.1, you were given the angle measure of 30° and then you determined the trigonometric function values. For each of the following examples, you will be given a point P on the terminal side of the angle.

**Example II.B.2:** Suppose that the point (-1, 1) lies on the terminal side of an angle θ in standard position. Find the six trigonometric function values of the angle.

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex2-b2.html)

**Example II.B.3:** Suppose that the point (–1, –square root of negative 3) lies on the terminal side of an angle θ in standard position. Find sin θ, cos θ, and tan θ.

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex2-b3.html)

**Example II.B.4:** Suppose that the point (6, –8) lies on the terminal side of an angle θ in standard position. Find sin θ, cos θ, and tan θ.

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex2-b4.html)

As illustrated in the preceding examples, the signs of the trigonometric function values depend on the quadrant of the angle.

Focusing on the positive values, if you remember the sequence all-sin-tan-cos, then you can quickly recall how the positive values correspond to the quadrants. This is illustrated in the circle diagram below.

|  |  |
| --- | --- |
| quadrant chart |  |

##### Coterminal Angles

If two angles are coterminal, then their trigonometric function values are the same, because the reference triangles are the same. For example, sin 30° = sin 390° = sin(–330°).

#### C. Trigonometric Function Values of Multiples of 90°

For the special cases where an angle is a multiple of 90°, there is no associated reference triangle. The definitions of the trigonometric functions still apply, but some of the trigonometric values are undefined, due to denominators that are equal to 0. The following table shows how to determine the values of the sine, cosine, and tangent for the angles of 0°, 90°, 180°, and 270°. In each case, the point P chosen on the terminal side of the angle is a point on the unit circle.

|  |  |
| --- | --- |
| **Angle** | **Trigonometric Function Values** |
|  |  |
|  |  |
|  |  |
|  | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/TRV4.gif |

#### D. Symmetry and Reference Angles

Recall that commonly occurring acute angles are 30°, 45°, and 60°. Commonly occurring angles between 0° and 360° include 30°, 45°, 60°, 120°, 135°, 150°, 210°, 225°, 240°, 300°, 315°, and 330°. Fortunately, if you know the trigonometric values for the acute angles, then by using symmetry, you can easily derive the exact trigonometric values for all of the angles on this list, without using a calculator!

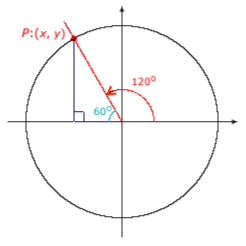
**Example II.D.1:** Find sin 120° and tan 120°.

**Solution:**

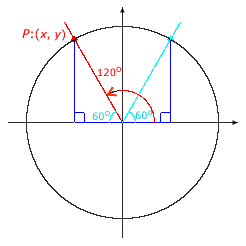
120° is a quadrant II angle.

The reference triangle has a vertex at the origin.

The angle at that vertex is 180° – 120° = 60°. This acute angle is called the reference angle.



Reflect the reference triangle across the y-axis to get a triangle in the first quadrant.



The reference triangle associated with 120° is congruent to the quadrant I reference triangle associated with 60°. The trigonometric function values associated with the quadrant II triangle are the same as the values associated with the quadrant I triangle, except for possibly the signs.

In quadrant II, the sine is positive. Therefore, sin 120° = sin 60° = square root of negative 3/2.  
In quadrant II, the tangent is negative. Therefore, tan 120° = –tan 60° = –square root of negative 3.

The method used in this example relies on the idea of a reference angle.

# Reference Angle

Given an angle θ in standard position, find the associated acute angle sigma formed by the terminal side and the x-axis. The angle sigma is called the reference angle. In the reference triangle associated with θ, the reference angle is the acute angle whose vertex is the origin.

The reference triangle associated with angle θ is congruent to the quadrant I reference triangle associated with angle sigma. The trigonometric function values of the angle θ are the same as the corresponding trigonometric function values of the angle sigma, except for possibly the signs. The signs are determined by the quadrant of the angle θ.

Coterminal angles, since they have the same associated reference triangle, have the same reference angle. The calculation of the measure of the reference angle a depends upon the quadrant of the given angle, as summarized in the following table:

|  |  |
| --- | --- |
| **Calculation of the Measure of the Reference Angle** | |
| Step 1: Given angle theta, find a coterminal angle  having measure between 0° and 360°.  Step 2: Locate the quadrant associated with θ and .  Step 3: Determine the reference angle sigma as shown below. | |
| Angles θ and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi.gif in Quadrant II | Angles θ and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi.gif in Quadrant I |
| Reference Angle sigma = 180° – | Reference Angle sigma = |
| Angles θ and  in Quadrant III | Angles θ  and  in Quadrant IV |
| Reference Angle  =  – 180° | Reference Angle  = 360° – https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi.gif |

**Example II.D.2:** Find cos 210°.

**Solution:**

|  |  |
| --- | --- |
| 210° is a quadrant III angle whose measure is between 0° and 360°.  Consulting the calculation table for a quadrant III angle, determine that the measure of the reference angle is 210° – 180° = 30°. |  |
| It is not imperative that you memorize the formulas for calculating the reference angle. You can draw the reference triangle associated with 210° and see that the acute angle at the origin in the triangle has measure 30°. The reference triangle associated with 210° is congruent to the quadrant I reference triangle associated with 30°, so the cosine of 210° must be the same as the cosine of 30°, except for possibly the sign. |  |
| In quadrant III, cosine is negative, so cos 210° = –cos 30° = –square root of negative 3/2. | |

**Example II.D.3:** Find cos 1035°.

**Solution:**

First find an angle that is coterminal with 1035° and has measure between 0° and 360°.

Subtract 360° to get a coterminal angle: 1035° – 360° = 675° but is not between 0° and 360°.

Subtract 360° again: 675° – 360° = 315°.

|  |  |
| --- | --- |
| 315° is a quadrant IV angle that is coterminal with 1035° and has measure between 0° and 360°. Since 1035° and 315° are coterminal, cos 1035° = cos 315°.  Consulting the calculation table for a quadrant IV angle, you'll find that the reference angle is 360° – 315° = 45°. |  |
| It is not imperative that you memorize the formulas for calculating the reference angle. You can draw the reference triangle associated with 315° and see that the acute angle at the origin in the triangle has measure 45°. The reference triangle associated with 315° is congruent to the quadrant I reference triangle associated with 45°, so the cosine of 315° must be the same as the cosine of 45°, except for possibly the sign. |  |

In quadrant IV, cosine is positive, so cos 1035° = cos 315° = cos 45° = square root of 2/2.

[*Return to top of page*](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/Trigonometric%20Functions%20of%20Any%20Angle.html#pagetop)

[**Report broken links or any other problems on this page.**](http://help.umuc.edu/)  
  
[**Copyright © by University of Maryland University College.**](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/common/copyright.html)